Sobolev and *BV* functions on metric measure spaces [1]

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[1] Papers in collaboration with N.Gigli, G.Savaré, M.Colombo, S.Di Marino

A.-Gigli-Savaré: *Density of Lipschitz maps and weak gradients in metric measure spaces.* (Revista Matematica Iberoamericana, 2013)

We study different notions of (modulus of) gradient in metric measure spaces $(X, d, \boldsymbol{m})$. We use the theory of gradient flows in Hilbert spaces and ideas coming from optimal mass transportation to show the equivalence of these notions.

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Motivations: fractals, Gromov-Hausdorff limits of Riemannian manifolds, graphs, the space $\mathscr{P}(X)$, spaces with singularities,....

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2. Sobolev spaces in metric measure spaces $(X, d, \boldsymbol{m})$
3. Identification of gradients
4. The spaces $BL^{1,1}$ and BV

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Sobolev spaces in $\mathbb{R}^n$

The Sobolev spaces $H^{1,q}(\mathbb{R}^n) = W^{1,q}(\mathbb{R}^n)$, $1 < q < \infty$, can be defined by:

(approximation) $f \in H^{1,q}(\mathbb{R}^n)$ if $f \in L^q(\mathbb{R}^n)$ and there exist smooth f_h convergent to f in $L^q(\mathbb{R}^n)$ such that

$$\limsup_{h\to\infty} \int_{\mathbb{R}^n} |\nabla f_h|^q \, dx < \infty.$$

(integration by parts) $f \in W^{1,q}(\mathbb{R}^n)$ if $f \in L^q(\mathbb{R}^n)$ and there exists $\nabla f \in L^q(\mathbb{R}^n; \mathbb{R}^n)$ such that

$$\int_{\mathbb{R}^n} f \nabla \phi \, dx = -\int_{\mathbb{R}^n} \phi \nabla f \, dx \qquad \forall \phi \in C_c^\infty(\mathbb{R}^n).$$

Meyers-Serrin (1960), "$H = W$", by mollifications.

While the first definition can be adapted to metric measure spaces, the second one seems to require requires a certain "differentiable structure" (Riemannian manifolds, Gaussian spaces, Carnot-Carathéodory spaces,...).

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Sobolev spaces on manifolds

For instance, on a Riemannian manifold (M, g), a coordinate-free definition is based on the integration by parts formula (which could be taken as definition of the Riemannian divergence)

$$\int_M f \mathrm{div}\, X \, d\mathrm{vol}_M = -\int_M g(\nabla f, X) \, d\mathrm{vol}_M$$

for any smooth compactly supported section X of TM and any $f \in C^1(M)$, so that $f \in L^q(\mathrm{vol}_M)$ belongs to $W^{1,q}(M)$ if there exists a section, denoted ∇f, of TM with $g(\nabla f, \nabla f) \in L^{q/2}(\mathrm{vol}_M)$ and

$$\int_M f \mathrm{div}\, X \, d\mathrm{vol}_M = -\int_M g(\nabla f, X) \, d\mathrm{vol}_M$$

for any smooth compactly supported section X of TM.

Levi's approach

B.Levi, *Sul principio di Dirichlet*, Rend. Circ. Mat. Palermo, 1906.

f belongs to $BL^{1,q}(\mathbb{R}^n)$ if $\int_{\mathbb{R}^n}|f|^q\,dx<\infty$ and, for all $i\in\{1,\ldots,n\}$, it holds:

(i) for $\mathscr{L}^{n-1}$-a.e. $x_i'\in\mathbb{R}^{n-1}$ the map $x_i\mapsto f(x_i,x_i')$ is locally absolutely continuous in $\mathbb{R}$;

(ii) $\int_{\mathbb{R}^{n-1}}\int_{\mathbb{R}}|\frac{d}{dx_i}f(x_i,x_i')|^q\,dx_i dx_i'<\infty$.

Using the derivative of one-dimensional restrictions and Fubini's theorem it is still possible "componentwise" to define a gradient ∇f.

Theorem. $BL^{1,q}(\mathbb{R}^n)\subset W^{1,q}(\mathbb{R}^n)$. *Conversely, if $f\in H^{1,q}(\mathbb{R}^n)$ there exists $\tilde f\in BL^{1,q}(\mathbb{R}^n)$ $\mathscr{L}^n$-a.e. equal to f.*

Summing up, BL is intermediate between W and H, but a posteriori the 3 spaces coincide. One may take for instance $\tilde f(x)=\limsup_{\epsilon\downarrow 0} f*\rho_\epsilon(x)$.

This problem has been studied, even for BV functions, by Caccioppoli, Cesari, Vol'pert, Federer.

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Sobolev spaces in metric measure spaces $(X, d, \boldsymbol{m})$

Let us consider a complete and separable metric space (X, d). Given $f : X \to \mathbb{R}$, we define slope of f at x (also called local Lipschitz constant) by

$$|\nabla f|(x) := \limsup_{y \to x} \frac{|f(y) - f(x)|}{d(y, x)}.$$

If we have a reference measure $\boldsymbol{m} \in \mathscr{P}(X)$, we say that $f \in L^q(X, \boldsymbol{m})$ belongs to $H^{1,q}(X, d, \boldsymbol{m})$ if there exist $f_h \in \mathrm{Lip}(X)$ such that $f_h \to f$ in $L^q(X, \boldsymbol{m})$ and

$$\limsup_{h \to \infty} \int_X |\nabla f_h|^q \, d\boldsymbol{m} < \infty.$$

In order to obtain a modulus of gradient, we call (following Cheeger) *relaxed slope* any possible weak limit in $L^q(X, \boldsymbol{m})$ of $|\nabla f_h|$.

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First differential calculus with $|\nabla f|_{*,q}$

Theorem. *The collection of q-relaxed slopes is a convex closed set. Its element with minimal L^q norm is denoted by $|\nabla f|_{*,q}$. Furthermore, we have the improved approximation*

$$|\nabla f|_{*,q} = \lim_{h\to\infty} |\nabla f_h| \quad \textit{strongly in } L^q(X,\boldsymbol{m}) \textit{ for some } f_h \in \mathrm{Lip}(X).$$

Starting from this, the first bits of differential calculus can be developed:

(chain rule) $|\nabla \phi(f)|_{*,q} \le |\phi'(f)||\nabla f|_{*,q}$, with equality if $\phi' \ge 0$;

(pointwise minimality) $|\nabla f|_{*,q} \le h$ $\boldsymbol{m}$-a.e. in X for any relaxed slope h of f.

(locality) $|\nabla f|_{*,q} = |\nabla g|_{*,q}$ $\boldsymbol{m}$-a.e. in $\{f = g\}$.

For instance, locality can be proved first by a reduction to the case $g = 0$, and then choosing $\phi_\epsilon(t) = (t-\epsilon)^+$, with $\epsilon \to 0^+$, using the variational characterizazion of $|\nabla f|_{*,q}$.

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The "Dirichlet" energy Ch

From now on we assume for simplicity $q = 2$ and define $\mathrm{Ch} : L^2(X, \boldsymbol{m}) \to [0, \infty]$ by

$$\mathrm{Ch}(f) := \inf \left\{ \liminf_{h\to\infty} \int_X |\nabla f_h|^2 \, d\boldsymbol{m} : \ \|f_h - f\|_2 \to 0, \ \ f_h \in \mathrm{Lip}(X) \right\}.$$

Theorem (Integral representation of Ch)

$\mathrm{Ch}(f) < \infty$ *if and only if* $f \in H^{1,2}(X, d, \boldsymbol{m})$ *and*

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From the stability properties of relaxed slopes, or directly from the definition, one obtains that Ch is a convex lower semicontinuous functional in $L^2(X, \boldsymbol{m})$ with dense domain (because it includes $\mathrm{Lip}(X)$).

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then it is easily seen that

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As a consequence, Ch is 2-homogeneous but not quadratic! Metric measure spaces such that Ch is a quadratic form are called *infinitesimally Hilbertian*.

Roughly speaking, this marks the same difference between Riemannian and Finsler manifolds.

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The Laplacian

Since Ch is convex and lower semicontinuous, we may define the subdifferential

$$\partial \mathrm{Ch}(f) := \left\{ \xi \in L^2(X, \boldsymbol{m}) : \ \mathrm{Ch}(g) \geq \mathrm{Ch}(f) + \int_X \xi (g - f) \, d\boldsymbol{m} \ \ \forall g \right\}$$

and, having in mind the classical case, define $-\Delta f$ as the element with minimal norm in $\partial \frac{1}{2}\mathrm{Ch}(f)$.

The Hilbertian theory of gradient flows then provides a continuous semigroup $P_t g$ in $L^2(X,)$ satisfying

$$\frac{d}{dt} P_t g = \Delta P_t g \qquad \text{for } \mathscr{L}^1\text{-a.e. } t > 0.$$

We shall call it the L^2 heat flow.

In general the laplacian is a nonlinear operator: it is linear iff the metric measure structure is infinitesimally Hilbertian. The same holds for P_t.

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(Heinonen-Koskela) We say that a Borel function g is an upper gradient for $f : X \to \mathbb{R}$ if

$$\left|f(\gamma_1) - f(\gamma_0)\right| \leq \int_\gamma g$$

for any absolutely continuous curve $\gamma : [0, 1] \to X$. The concept makes sense, since absolute continuity is a metric notion!

More precisely, the curvilinear integral makes sense thanks to the formula

$$\int_\gamma g := \int_0^1 g(\gamma_s)|\dot{\gamma}_s|\, ds$$

and it retains its usual invariance properties. The quantity

$$|\dot{\gamma}_t| := \lim_{h \to 0} \frac{d(\gamma_{t+h}, \gamma_t)}{|h|}$$

is the so-called *metric derivative* of γ.

Can we pass from the metric derivative to the modulus of gradient?

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The reference measure $\boldsymbol{m}$ comes into play if we look for the "smallest possible g". Already the theory in Euclidean spaces tells us that it would not be realistic to impose the absolute continuity property along *all* curves.

Notation. We denote by $e_t : C([0,1];X) \to X$ the evaluation map at time $t \in [0,1]$, namely $e_t(\gamma) := \gamma(t)$.

Definition. [AGS] $\pi \in \mathscr{P}(C([0,1];X))$ is said to be a test plan if π is concentrated on $AC([0,1];X)$ and

(1) $\int \int_0^1 |\dot\gamma_t|^2 \, dt \, d\pi(\gamma) < \infty$;

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Definition. We say that a property holds for a.e. curve if the set where the property fails is π-negligible for any test plan π.

This property can be compared with Fuglede's potential theoretic notion for non-parametric curves, based on 2-modulus, see A-Di Marino-Savaré (JEMS, 2005).

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Definition. [AGS] $\boldsymbol{\pi} \in \mathscr{P}(C([0,1];X))$ is said to be a test plan if $\boldsymbol{\pi}$ is concentrated on $AC([0,1];X)$ and

(1) $\int \int_0^1 |\dot{\gamma}_t|^2 \, dt \, d\boldsymbol{\pi}(\gamma) < \infty$;

(2) there exists $C = C(\boldsymbol{\pi}) \geq 0$ such that

$$(e_t)_\sharp \boldsymbol{\pi} \leq C\boldsymbol{m} \qquad \text{for all } t \in [0,1].$$

Definition. We say that a property holds for a.e. curve if the set where the property fails is $\boldsymbol{\pi}$-negligible for any test plan $\boldsymbol{\pi}$.

This property can be compared with Fuglede's potential theoretic notion for non-parametric curves, based on 2-modulus, see A-Di Marino-Savaré (JEMS, 2005).

Levi's approach in metric measure spaces

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Roles of distance and measure

The condition $(e_t)_\sharp \boldsymbol{\pi} \leq C\boldsymbol{m}$, equivalent to

$$\max_{t\in[0,1]} \int \psi(\gamma_t)\, d\boldsymbol{\pi}(\gamma) \leq C \int_X \psi\, d\boldsymbol{m} \qquad \forall \psi \in C_b(X),\ \psi \geq 0$$

is a quantitative version of the fact that the curves, at any given time t, do not "concentrate" too much *relative* to $\boldsymbol{m}$.

Notice also that while Ch "mixes" in a nontrivial way distance and measure, in Levi's approach (as well as in Lott-Sturm-Villani) the roles of d and $\boldsymbol{m}$ are better decoupled (the former is used to build the distance W_2, the latter to build the entropies).

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Definition. [AGS] (Fuglede, Koskela-MacManus). We say that g is a weak upper gradient of f, and write $g \in WUG(f)$, if

$$\big|f(\gamma_1) - f(\gamma_0)\big| \le \int_\gamma g \qquad \text{for a.e. curve } \gamma.$$

Weak upper gradients have also a nice stability property:

$$f_n \to f \ \boldsymbol{m}\text{-a.e. in } X,\ g_n \in WUG(f_n),\ g_n \rightharpoonup g \quad \Longrightarrow \quad g \in WUG(f).$$

One reduces to the case when $g_n \to g$ in $L^q(X, \boldsymbol{m})$, and even $\sum_n \|g_n - g\|_2 < \infty$. Then one uses the principle (Fuglede)

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The gradient $|\nabla f|_w$

In particular $WUG(f)$ is a convex closed subset of $L^2(X, \boldsymbol{m})$. The element with minimal L^2 norm will be denoted by $|\nabla f|_w$.
Another consequence of stability is the inequality

$$(*) \qquad |\nabla f|_w \le |\nabla f|_* \qquad \boldsymbol{m}\text{-a.e. in } X.$$

Essentially it follows by the fact that $|\nabla f|_*$, being the weak (even strong) limit of $|\nabla f_n|$ with f_n Lipschitz, belongs to $WUG(f)$.

(*) is the infinitesimal version, in this metric context, of the "easy" inclusion $H^{1,2} \subset BL^{1,2}$.

Proving the converse inequality requires the construction of Lipschitz approximating functions f_h, with $\int |\nabla f_h|^2 \, d\boldsymbol{m}$ uniformly bounded, starting from the only informations that f is "$W^{1,2}$ along almost all curves" and that $|\nabla f|_w \in L^2(X, \boldsymbol{m})$.

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A classical (but finite-dimensional) strategy

If we assume $\boldsymbol{m}$ to be doubling (i.e. $\boldsymbol{m}(B_{2r}(x)) \le c_D \boldsymbol{m}(B_r(x))$ for all balls $B_r(x)$) and the validity of the local Poincaré inequality

$$\text{(PI)} \qquad \int_{B_r(x)} |f - f_{r,x}| \, d\boldsymbol{m} \le cr \int_{B_{\lambda r}(x)} g \, d\boldsymbol{m}$$

for $g \in WUG(f)$, then a family of approximating Lipschitz functions can be obtained starting from the maximal function of g

$$M_g(x) := \sup_{r>0} \frac{1}{\boldsymbol{m}(B_r(x))} \int_{B_r(x)} g(y) \, d\boldsymbol{m}(y),$$

which belongs to $L^2(X, \boldsymbol{m})$, and noticing that the restriction of f to the sets $\{M_g \le \lambda\}$ is Lipschitz, with Lipschitz constant $C\lambda$.
By extending these restrictions one obtains Lipschitz approximating functions, not only in Sobolev norm, but also in the Lusin sense.

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The identification theorem

Theorem. [AGS] *Let* (X, d) *be complete and separable,* $\boldsymbol{m}$ *Borel nonnegative measure, finite on bounded sets,* $1 < q < \infty$*. Then* $H^{1,q} = BL^{1,q}$ *and*

$$|\nabla f|_{*,q} = |\nabla f|_{w,q} \qquad \boldsymbol{m}\text{-a.e. in } X.$$

Tools:

• Gradient flows and sharp estimates on the dissipation of suitable "entropies";

• Optimal transport theory;

• Lifting of solutions to the continuity equation

$$\frac{d}{dt}\mu_t + \nabla \cdot (v_t \mu_t) = 0$$

to probabilities π on paths (all properly understood in a metric setup).

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The new strategy

We use the gradient flow of $\frac{1}{2}$Ch. This idea is natural, since this flow plays the role of convolution in this “nonlinear” context.
We compute the dissipation of the entropy

$$t \mapsto \int g_t \log g_t \, d\boldsymbol{m}$$

along the gradient flow of $\frac{1}{2}$Ch in two conceptually different ways.
The first formula uses just the Hilbertian formalism:

$$-\frac{d}{dt}\int_X g_t \log g_t \, d\boldsymbol{m} = -\int_X \log g_t \Delta g_t \, d\boldsymbol{m} = \int_X \frac{|\nabla g_t|_*^2}{g_t} \, d\boldsymbol{m}.$$

The second equality comes with an integration by parts that can be proved with the “subdifferential” definition of laplacian, and even when Δ is not linear. The quantity on the right hand side is the so-called Fisher information functional.

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Using weak upper gradients, a sharper estimate of entropy dissipation can be given:

$$-\frac{d}{dt}\int_X g_t \log g_t \, d\boldsymbol{m} \le \frac{1}{2}\int_X \frac{|\nabla g_t|_*^2}{g_t}\, d\boldsymbol{m} + \frac{1}{2}\int_X \frac{|\nabla g_t|_w^2}{g_t}\, d\boldsymbol{m}.$$

By comparing the two we get

$$4\int_X |\nabla\sqrt{g_t}|_*^2 \, d\boldsymbol{m} = \int_X \frac{|\nabla g_t|_*^2}{g_t}\, d\boldsymbol{m} \le \int_X \frac{|\nabla g_t|_w^2}{g_t}\, d\boldsymbol{m} = 4\int_X |\nabla\sqrt{g_t}|_w^2 \, d\boldsymbol{m}.$$

Then, if we assume that f stays between two positive constants and we apply the gradient flow with initial condition $g := f^2$, by letting $t \to 0$ we achieve the result.

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The sharp energy dissipation estimate

Since this estimate involves the derivative along curves, we need to connect “Eulerian” and “Lagrangian” viewpoints. The tools are two: the superposition principle and Kuwada's lemma.

Recall that the Wasserstein distance $W_2^2(\mu,\nu)$ (possibly infinite) between $\mu,\nu \in \mathscr{P}(X)$ is defined by the minimum value in the optimal transport problem

$$\min\left\{\int_{X\times X} d^2(x,y)\,d\Sigma(x,y):\ (\pi_1)_\sharp\Sigma=\mu,\ (\pi_2)_\sharp\Sigma=\nu\right\}$$

in Kantorovich's formulation (1941) of the problem originally raised by Monge (1781). Here π_i, $i=1,2$, are the canonical projections on the factors, hence the constraint on Σ can be written as

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$$\Sigma(A\times X)=\mu(A)\quad\forall A\in\mathscr{B}(X),\qquad \Sigma(X\times B)=\nu(B)\quad\forall B\in\mathscr{B}(X).$$

The sharp energy dissipation estimate

Since this estimate involves the derivative along curves, we need to connect "Eulerian" and "Lagrangian" viewpoints. The tools are two: the superposition principle and Kuwada's lemma.
Recall that the Wasserstein distance $W_2^2(\mu,\nu)$ (possibly infinite) between $\mu,\ \nu \in \mathscr{P}(X)$ is defined by the minimum value in the optimal transport problem

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The superposition principle

The superposition principle (L.C.Young, Smirnov, [AGS]) asserts that any curve of measures $t \in [0, T] \mapsto \mu_t \in \mathscr{P}(\mathbb{R}^n)$ satisfying the continuity equation

$$(*) \qquad \frac{d}{dt}\mu_t + \mathrm{div}\big(v_t\mu_t\big) = 0$$

with $\|v_t\|_{L^2(\mu_t)} \in L^1$ is absolutely continuous with respect to W_2 and representable as superposition of curves, i.e., for some $\pi \in \mathscr{P}(C([0, T]; \mathbb{R}^n))$ it holds $(e_t)_\sharp\pi = \mu_t$ for all $t \in [0, T]$. Moreover

$$(**) \qquad |\dot\mu_t|^2 \leq \int_{\mathbb{R}^n} |v_t|^2 \, d\mu_t \qquad \text{for a.e. } t$$

and there exists an "optimal" v_t satisfying (*) for which (**) holds with equality.

This principle has been widely used in [AGS] to formalize Otto's calculus and the Riemannian structure of $\mathscr{P}_2(\mathbb{R}^n)$. The metric version is due to Lisini.

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Theorem. (Lisini) *Let $t \in [0,T] \mapsto \mu_t \in \mathscr{P}(X)$ be an AC^2 curve. Then there exists $\pi \in \mathscr{P}(C([0,T];X))$, concentrated on $AC^2([0,T];X)$, such that $(e_t)_\sharp \pi = \mu_t$ for all $t \in [0,T]$ and*

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We are going to apply this principle to the curve $\mu_t = g_t \boldsymbol{m}$, with g_t gradient flow of $\frac{1}{2}$Ch starting from $g_0 \in L^\infty(X, \boldsymbol{m})$ nonnegative e normalized ($\int g_0 \, d\boldsymbol{m} = 1$).

Whenever $\boldsymbol{m}(B_r(\bar x)) \leq Ce^{cr^2}$, these properties are preserved in time, hence $\mu_t \in \mathscr{P}(X)$ and the probability π given by Lisini's theorem is a test plan (i.e. it satisfies the non-concentration condition), having marginals with bounded density.

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Kuwada's lemma

Lemma. *Let* $g_0 \in L^2(X, \boldsymbol{m})$ *and let* (g_t) *be the gradient flow of* $\frac{1}{2}\mathrm{Ch}$ *starting from* g_0*. Assume that* $\int_X g_0 \, d\boldsymbol{m} = 1$*. The the curve* $t \mapsto \mu_t := g_t \boldsymbol{m} \in \mathscr{P}(X)$ *is absolutely continuous with respect to* W_2 *and it holds*

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The proof exploits the deep link between optimal transport and the Hopf-Lax semigroup

$$Q_t\varphi(x) := \inf_{y \in X} \varphi(y) + \frac{d^2(x, y)}{2t}.$$

In particular we use the (metric) subsolution property: for any $x \in X$, with at most countably many exceptional times, it holds:

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We use the duality formula and interpolation to write ($s < t$):

$$\begin{aligned}\frac{1}{2}W_2^2(\mu_s, \mu_t) &= \sup_\phi -\int_X \phi\, d\mu_s + \int_X Q_1\phi\, d\mu_t \\ &= \sup_\phi \int_0^1 \frac{d}{d\ell}\int_X Q_\ell\phi\, d\mu_{s+\ell(t-s)}\, d\ell.\end{aligned}$$

By the Leibniz rule and the subsolution property, the supremum can be estimated from above with

$$\int_0^1 \int_X -\frac{1}{2}|\nabla Q_\ell\phi|^2 g_{s+\ell(t-s)}\, d\boldsymbol{m} d\ell + (t-s)\int_0^1 \int_X Q_\ell\phi \Delta g_{s+\ell(t-s)}\, d\boldsymbol{m} d\ell.$$

Eventually Young inequality and integration by parts give the estimate

$$\frac{1}{2}(t-s)^2 \int_0^1 \int \frac{|\nabla g_{s+\ell(t-s)}|^2}{g_{s+\ell(t-s)}}\, d\boldsymbol{m}\, d\ell$$

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Proof of the sharp energy dissipation estimate

Let $\pi \in \mathscr{P}(C([0,1];X))$ be a test plan associated to $\mu_t = g_t \boldsymbol{m}$, given by the metric superposition principle. Using the convexity of $z \log z$ and the chain rule $|\nabla \log(g_t)|_w = |\nabla g_t|_w / g_t$ we get

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Sobolev spaces via integration by parts

Can we recover the "integration by parts" point of view also on metric measure spaces?

Definition. (Weaver) A derivation is a linear functional $\boldsymbol{b} : \mathrm{Lip}_b(X) \to L^0(X, \boldsymbol{m})$ satisfying the Leibniz rule.

Divergence. $\int_X \boldsymbol{b}(f)\, d\boldsymbol{m} = -\int_X f \mathrm{div}\, \boldsymbol{b}\, d\boldsymbol{m}$.

Norm $|\boldsymbol{b}|$ of a derivation. The least g satisfying $|\boldsymbol{b}(f)| \leq g|\nabla f|$ $\boldsymbol{m}$-a.e. in X for all $f \in \mathrm{Lip}_b(X)$.

Derivations play the role of vector fields in this theory (see Gigli's second Memoirs for much more).

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Definition. Let $1 < q < \infty$, $p = q'$. We say that $f \in L^p(X, \boldsymbol{m})$ belongs to $W^{1,p}(X, \boldsymbol{m})$ if there exists a linear functional L_f on L^q derivations with divergence in L^q such that

$$\int_X L_f(\boldsymbol{b})\, d\boldsymbol{m} = -\int_X f \mathrm{div}\, \boldsymbol{b}\, d\boldsymbol{m} \quad \forall \boldsymbol{b} \text{ with } |\boldsymbol{b}| + |\mathrm{div}\, \boldsymbol{b}| \in L^q(X, \boldsymbol{m}).$$

It can be proved (Di Marino) that

$$H^{1,p}(X, d, \boldsymbol{m}) \subset W^{1,p}(X, d, \boldsymbol{m}) \subset BL^{1,p}(X, d, \boldsymbol{m}).$$

As a consequence, the identification theorem tells us that the 3 spaces coincide, with essentially no assumption on the metric measure structure!

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The spaces $BL^{1,1}$ and $H^{1,1}$

The proof of the identification theorem fails in the case $q = 1$, by the lack of semicontinuity of $f \mapsto \int_X |\nabla f| \, d\boldsymbol{m}$, even in nice spaces.

3 possible definitions:

• One could define $BL^{1,1}(X, \mathrm{d}, \boldsymbol{m})$ via weak upper gradients (using 1-test plans)
• One could define $H^{1,1}(X, \mathrm{d}, \boldsymbol{m})$ considering approximating sequences (f_n) for which $|\nabla f_n|$ are $\boldsymbol{m}$-equiintegrable;
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In general, the equivalence between these definitions fails (they coincide under doubling & Poincaré).

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The space *BV*

In the Euclidean space $\mathbb{R}^n$ (or other nice spaces), *BV* functions f are defined by the existence of a vector-valued measure with finite total variation

$$\mathbf{D}f = \big(D_1 f, \ldots, D_n f\big),$$

representing the derivative in the sense of distributions. In recent years, *BV* functions in infinite-dimensional spaces have been investigated, starting from the work of Fukushima in Gaussian (Wiener) spaces.

Definition. (Miranda Jr, 1996) Let $f \in L^1(X, \mathrm{d}, \boldsymbol{m})$. We say that $f \in BV(X, \mathrm{d}, \boldsymbol{m})$ if

$$|\mathbf{D}f|(X) := \inf\left\{\liminf_{n\to\infty} \int_X |\nabla f_n|\, d\boldsymbol{m} : \ f_n \to f \text{ in } L^1(X),\ f_n \in \mathrm{Lip}_{\mathrm{loc}}(X)\right\} < \infty.$$

Miranda proved that, for $f \in L^1_{\mathrm{loc}}(X)$, in locally compact spaces the set function $A \mapsto |\mathbf{D}f|(A)$ is always the restriction to open sets of X of a Borel (possibly infinite) measure, the so-called total variation measure.

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Is there an equivalent definition of *BV* and of $|\mathbf{D}f|$ based on (measure) weak upper gradients?

Does it work also in non locally compact situations?

Having in mind the *BV* 1-dimensional estimate (for nice functions f)

$$\begin{aligned} |f\circ\gamma(1) - f\circ\gamma(0)| &\leq |\mathbf{D}(f\circ\gamma)|(0,1) = \int_0^1 |\nabla f|(\gamma_t)|\gamma_t'|\,dt \\ &\leq \mathrm{Lip}(\gamma)\int_0^1 |\nabla f|(\gamma_t)|\,dt \end{aligned}$$

we may average the inequality w.r.t. γ and give the following "weak upper gradient" definition of the space *BV*.

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Definition. Let $f \in L^1(X, \boldsymbol{m})$, we say that $f \in BV_w(X, \mathsf{d}, \boldsymbol{m})$ if there exists a positive finite measure $\boldsymbol{\mu}$ in X satisfying

$$\int \gamma_\sharp |\mathbf{D}(f \circ \gamma)| \, d\boldsymbol{\pi}(\gamma) \leq C(\boldsymbol{\pi}) \|\mathrm{Lip}(\gamma)\|_{L^\infty(\boldsymbol{\pi})} \, \boldsymbol{\mu}$$

for all ∞-test plans $\boldsymbol{\pi}$. The minimal measure μ with this property will be denoted by $|\mathbf{D}f|_w$.

Theorem. (A-Di Marino, 2012) *$BV_w(X, \mathsf{d}, \boldsymbol{m}) = BV(X, \mathsf{d}, \boldsymbol{m})$ and $|\mathbf{D}f|_w = |\mathbf{D}f|$.*

The proof involves the gradient flow of $f \mapsto |\mathbf{D}f|(X)$ and suitable limiting versions as $p \to \infty$ of the tools (Hamilton-Jacobi equations, superposition principle) that we have seen in the Sobolev case.

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